

Common eigenfunctions of commuting differential operators of rank 2

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Introduction

Let us consider two differential operators

$$L_n = \sum_{i=0}^n u_i(x) \partial_x^i, \quad L_m = \sum_{i=0}^m v_i(x) \partial_x^i.$$

If L_n and L_m commute, then there is a nonzero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$ (see [1]). The curve Γ defined by $R(z, w) = 0$ is called the *spectral curve*. If

$$L_n \psi = z \psi, \quad L_m \psi = w \psi,$$

then $(z, w) \in \Gamma$. For almost all $(z, w) \in \Gamma$ the dimension of the space of common eigenfunctions ψ is the same. The dimension of the space of common eigenfunctions of two commuting differential operators is called the *rank*. The rank is a common divisor of m and n .

If the rank equals 1, then there are explicit formulas for coefficients of commutative operators in terms of Riemann theta-functions (see [2]).

The case when rank is greater than one is much more difficult. The first examples of commuting ordinary scalar differential operators of the nontrivial ranks 2 and 3 and the nontrivial genus $g=1$ were constructed by Dixmier [8] for the nonsingular elliptic spectral curve $w^2 = z^3 - \alpha$, where α is arbitrary nonzero constant:

$$L = (\partial_x^2 + x^3 + \alpha)^2 + 2x, \\ M = (\partial_x^2 + x^3 + \alpha)^3 + 3x\partial_x^2 + 3\partial_x + 3x(x^2 + \alpha),$$

where L and M is the commuting pair of the Dixmier operators of rank 2, genus 1. There is an example

$$L = (\partial_x^3 + x^2 + \alpha)^2 + 2\partial_x, \\ M = (\partial_x^3 + x^2 + \alpha)^3 + 3\partial_x^4 + 3(x^2 + \alpha)\partial_x + 3x,$$

where L and M is the commuting pair of the Dixmier operators of rank 3, genus 1.

The general classification of commuting ordinary differential operators of rank greater than 1 was obtained by Krichever [3]. The general form of commuting operators of rank 2 for an arbitrary elliptic spectral curve was found by Krichever and Novikov [4]. The general form of operators of rank 3 for an arbitrary elliptic spectral curve was found by Mokhov [5], [6]. Mironov in [7] constructed examples of operators

$$L = (\partial_x^2 + A_3x^3 + A_2x^2 + A_1x + A_0)^2 + g(g+1)A_3x,$$

$$M^2 = L^{2g+1} + a_{2g}L^{2g} + \dots + a_1L + a_0,$$

where a_i are some constants and A_i , $A_3 \neq 0$, are arbitrary constants. Operators L and M are commuting operators of rank 2, genus g . Examples of commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients were constructed in [11] by Mokhov.

It is proved in [12] that the operators

$$L = (\partial_x^2 + Ax^6 + Bx^2)^2 + 16g(g+1)Ax^4,$$

$$M^2 = L^{2g+1} + a_{2g}L^{2g} + \dots + a_1L + a_0,$$

where A, B are arbitrary constants, $A \neq 0$, a_i are some constants, are commuting operators of rank 2.

In this paper we find common eigenfunctions of L and M . Until now common eigenfunctions of commuting differential operators with polynomial coefficient were not found explicitly.

The author is grateful to O.I.Mokhov for valuable discussions.

Commuting operators of rank 2

Consider the operator

$$L = (\partial_x^2 + V(x))^2 + W(x).$$

We know that ([7]) the operator commutes with an operator M of order $4g+2$ with hyperlyptic spectral curve of genus g and hence is operator of rank 2, if and only if there is a polynomial

$$Q = z^g + a_1(x)z^{g-1} + a_2(x)z^{g-2} + \dots + a_{g-1}(x)z + a_g(x)$$

that the following relation is satisfied

$$Q^{(5)} + 4VQ''' + 6V'Q'' + 2Q'(2z - 2W + V'') - 2QW' \equiv 0,$$

where Q' means $\partial_x Q$. The spectral curve has the form

$$4w^2 = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q''' + 2Q(2V'Q' + 4VQ'' + Q^{(4)}).$$

Common eigenfunctions of L and M satisfy the second order differential equation [7]

$$\psi''(x, P) - \chi_1(x, P)\psi'(x, P) - \chi_0(x, P)\psi(x, P) = 0,$$

where χ_0 and χ_1 have the form

$$\chi_1 = \frac{Q'}{Q}, \quad \chi_0 = -\frac{Q''}{2Q} + \frac{w}{Q} - V.$$

Common eigenfunctions of commuting differential operators of rank 2

Let me recall some definitions.

Bessel functions J_α are solutions of the Bessel equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0.$$

If α is not integer, then $J_\alpha, J_{-\alpha}$ satisfy Bessel equation, where

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha + 1)} \left(1 - \frac{x^2}{2^2 1! (\alpha + 1)} + \frac{x^4}{2^4 2! (\alpha + 2)} - \dots \right)$$

If α is integer, then $J_\alpha, J_{-\alpha}$ are not independent solutions. Note that (see [13])

$$J'_\alpha = \frac{\alpha J_\alpha(x)}{x} - J_{\alpha+1}(x).$$

Functions

$$Y_\alpha(x) = \frac{J_\alpha(x) \cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

are called Bessel functions of the second kind.

Solutions $H(\alpha, \lambda, \beta, \gamma, \delta, \eta; x)$ of the following equation

$$y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\alpha + \beta - \gamma - \delta + 1}{x-a} \right) y'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y(x) = 0.$$

are called Heun functions. This equation has four regular singular points $0, 1, a, \infty$. Confluent Heun equation is obtained from the Heun equation through a confluence process (see [14]), that is, a process where two singularities coalesce. Denote by $CH(\alpha, \beta, \gamma, \delta, \eta; x)$ the solution of the confluent Heun equation

$$y''(x) + \left(\frac{\beta + \gamma - \alpha + 2}{x - 1} + \frac{x\alpha}{x - 1} - \frac{\beta + 1}{x(x - 1)} \right) y'(x) + \left(\frac{\alpha(\beta + \gamma + 2) + 2\delta}{2(x - 1)} - \frac{\alpha(\beta + 1) - \beta(\gamma + 1) - 2\eta - \gamma}{2x(x - 1)} \right) y(x) = 0$$

where

$$y(0) = 1, \quad y'(0) = \frac{\beta(\gamma - \alpha + 1) + \gamma - \alpha + 2\eta}{2(\beta + 1)}.$$

There is formula for Bessel functions [14]

$$J_\alpha(x) = \frac{x^\alpha(2ix + 1)CH(1, 2\alpha, 1, 0, \frac{1}{2}; -2ix)}{\Gamma(\alpha + 1)2^\alpha e^{ix}}.$$

We know from [12] that

$$L = (\partial_x^2 + Ax^6 + Bx^4)^2 + 16g(g + 1)Ax^4$$

commutes with a differential operator M of order $4g + 2$. Let us assume that $B = 0$. If $g = 1$, then spectral curve of commuting pair L and M is equal to

$$w^2 = z(192A + z^2)$$

and differential equation for common eigenfunctions has the form

$$\psi'' - \frac{64Ax^3}{16Ax^4 + z}\psi' - \left(\frac{w - 96Ax^2}{16Ax^4 + z} - Ax^6 \right) \psi = 0. \quad (1)$$

Let us suppose that $w = 0$. So, $z = 0, \pm\sqrt{-192A}$. If $z = 0$, then solutions of (1) are

$$x^{\frac{5}{2}} J_{\frac{1}{8}}\left(\frac{x^4\sqrt{A}}{4}\right), \quad x^{\frac{5}{2}} Y_{\frac{1}{8}}\left(\frac{x^4\sqrt{A}}{4}\right).$$

If $z = \pm\sqrt{-192A}$, then solutions of (1) are

$$e^{-\frac{Ax^4}{4}} \sqrt{-\frac{1}{A}} CH\left(\frac{z}{32} \sqrt{-\frac{1}{A}}, -\frac{1}{4}, -2, 0, \frac{5}{4}; -\frac{16Ax^4}{z}\right),$$

$$xe^{-\frac{Ax^4}{4}} \sqrt{-\frac{1}{A}} CH\left(\frac{z}{32} \sqrt{-\frac{1}{A}}, \frac{1}{4}, -2, 0, \frac{5}{4}; -\frac{16Ax^4}{z}\right).$$

If $g = 2$, then spectral curve of commuting operators L and M is equal to

$$w^2 = z(20160A + z^2)(20736A + z^2).$$

If $z = 0$, then equation for common eigenfunctions has the form

$$(4Ax^8 + 35)\psi'' - 32Ax^7\psi' + (147Ax^6 + 4A^2x^{14})\psi = 0. \quad (2)$$

Equation (2) has solutions

$$\begin{aligned} CH(0, -\frac{1}{8}, -2, -\frac{35}{256}, \frac{387}{256}; -\frac{4Ax^8}{35}), \\ xCH(0, \frac{1}{8}, -2, -\frac{35}{256}, \frac{387}{256}; -\frac{4Ax^8}{35}). \end{aligned}$$

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